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DYNAMIC RESPONSE OF BEAMS ON GENERALISED ELASTIC FOUNDATIONS

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Abstract—Dynamic responses of beams on generalised elastic foundations is studied using the method of Initial parameters. The foundation model proposed by Vlasov and Leontev is modified by incorporating in the analysis the horizontal displacements in the elastic foundation thus making it more general and physically close to the actual situation. Results are compared with those reported by Rades, using Pasternak's foundation model and Winkler's model. The insufficiency of the Winkler's model in the study of dynamic responses (mainly the bending moments) is emphasized. Solutions presented are quite general for application to beams on generalised elastic foundations subjected to arbitrary external dynamic loads and (or) moments.

NOTATION

E_bI modulus of rigidity of the beam
E, ν elastic constants of the foundation medium
F, F_f auxiliary functions
H thickness of the foundation layer
$\{I_p\}$ matrix of initial parameters (Eq. (25))
$[K]$ matrix of Influence function (Eq. (25))
L characteric length (Eq. (18c)).
M bending moment of the beam
N_1 generalised shear
P concentrated dynamic load, $P_0e^{i\omega t}$
Q beam shear
$\{R\}$ matrix of responses (Eq. (25)).
T_j, S_h generalised forces (Eqns. (7))
U_i, W_k generalised displacements (Eqns. (1))
$\bar{W}, \bar{M}, \bar{Q}, \bar{q}$ dimensionless responses (Eqns. (42), (43))
W_c^*, M_c^* static deflection and bending moment at the centre of the beam
a_0 dimensionless frequency ratio (Eqns. (18))
b_0 dimensionless mass ratio (Eqns. (18))
g^* acceleration due to gravity
l half length of the beam
m_b mass of the beam per unit length of beam
p, q surface loads on the foundation
q_u uniformly distributed dynamic load, $q_o e^{i\omega t}$
q^* general dynamic load on the beam
u, w displacement components (Eqns. (1))
w_0 vertical deflection of the beam
x, y, z cartesian coordinates
t time variable
Ω frequency ratio (Eq. (44))
ω operating frequency
ϕ_i, ψ_k dimensionless functions (Eq. (1))
γ parameter depending on elastic properties of foundation
δ lateral thickness of the plane elastic body (Fig. 1)
$\eta = x/L$ dimensionless parameter
$\sigma_x, \sigma_z, \tau_{xz}$ stresses
ρ unit weight of foundation material
ρ_b unit weight of beam material
$\{\alpha\}$ non-homogeneous part of the responses (Eq. (26b))
$\lambda = \gamma L$ dimensionless parameter.

1. INTRODUCTION

The analysis of structures on elastic foundations is of considerable interest in several fields such as foundation engineering, aero-space structures etc. The earliest mathematical idealisation of a foundation medium was due to Winkler[1] which is a single parameters model. Kerr[2] has given a comprehensive review of the various foundation models generally used in these problems. The foundation models proposed by Vlasov and Leontev[3], Pasternak, Wieghardt, Filonenko-Borodich[2] are essentially two parameter models which have been reported to give better performance when compared to Winkler's model[2]. Among all the two parameter foundation models, Vlasov and leontev's model seems to be more advantageous because of the fact that the parameters characterising the medium have been expressed explicitly in terms of physically realisable material properties of the foundation medium. Starting with the elastic continuum equations, Kameswara Rao[4, 5] modified the foundation model proposed by Vlasov and Leontev[3], by incorporating in the analysis the horizontal displacements in the elastic foundation, thus making the model more general, physically close, and yet mathematically simple. Only static problems have been reported[4].

In a recent paper Rades[6] brought out the importance of the two parameter Pasternak's foundation model over the Winkler's model by studying the steady state dynamic responses of beams on such elastic foundations. He has shown, how, for the bending moments, the responses using the Pasternak's model are totally different from those obtained using Winkler's model, and observed that Winkler's model leads to large errors in the frequency range of practical interest. However he presented the results[6] neglecting the foundation inertia.

In the present investigation the dynamic responses of beams on generalised elastic foundations has been studied using the analytical model developed by the authors[4, 5]. Method of initial parameters has been presented thus facilitating the application of the solutions to problems involving arbitrary, external, dynamic loads and (or) moments. Results have been graphically compared with those reported by Rades[6]. A few typical results of the dynamic responses of the beam–foundation system, taking into account the foundation inertia also, have been graphically shown. "Geometric damping" implicitly exists in the present analysis because of the semi-infiniteness of the foundation medium.

2. BASIC EQUATIONS

Referring to the plane elastic body (Fig. 1) of thickness H (finite or infinite) and lateral thickness δ the displacements can be expressed as

$$u(x, z, t) = \sum_{i=1}^{m} U_i(x\, t)\phi_i(z) \tag{1}$$

$$w(x, z, t) = \sum_{k=1}^{n} W_k(x, t)\psi_k(z)$$

in which $\phi_i(z)$ and $\psi_k(z)$ are *a priori* chosen dimensionless functions consistent with the constraints of the problem[3, 4]; U_i, W_k are referred to as generalised displacements and t is the time variable.

Using Vlasov's general variational method[3] for the plane elasto-dynamic equations, the equations governing the generalised displacements can be obtained as[3, 5]

$$\sum_{i=1}^{m} a_{ij}U_i'' - \left(\frac{1-\nu}{2}\right)\sum_{i=1}^{m} b_{ji}U_i + \sum_{k=1}^{n}\left\{\nu t_{jk} - \frac{(1-\nu)}{2}c_{jk}\right\}W_k' + \frac{(1-\nu^2)}{E}p_j = 0(j = 1, 2 \ldots\ldots m) \tag{2a}$$

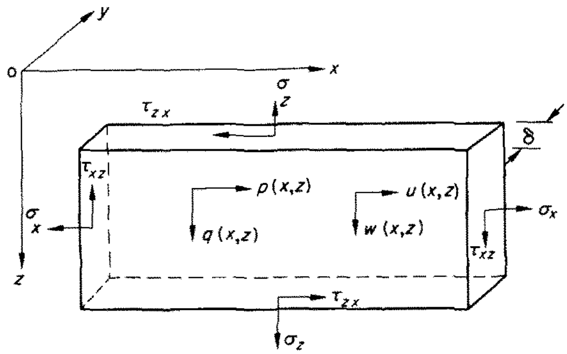

Fig. 1. Diagram showing the convention of stresses and displacements of plane elastic body.

$$-\sum_{i=1}^{m}\left\{\nu t_{hi}-\frac{(1-\nu)}{2}c_{hi}\right\}U'_i+\frac{(1-\nu)}{2}\sum_{k=1}^{n}r_{hk}W''_k-\sum_{k=1}^{n}s_{hk}W_k+\frac{(1-\nu^2)}{E}q_h=0\,(h=1,2,\ldots,n) \quad (2b)$$

where primes denote derivatives with respect to x and

$$a_{ij}=a_{ji}=\int_0^H \phi_j\phi_i\delta\,\mathrm{d}z;\; b_{ji}=b_{ij}=\int_0^H \frac{\mathrm{d}\phi_j}{\mathrm{d}z}\frac{\mathrm{d}\phi_i}{\mathrm{d}z}\delta\,\mathrm{d}z;$$
$$r_{hk}=r_{kh}=\int_0^H \psi_k\psi_h\delta\,\mathrm{d}z;\; s_{hk}=s_{kh}=\int_0^H \frac{\mathrm{d}\psi_k}{\mathrm{d}z}\frac{\mathrm{d}\psi_h}{\mathrm{d}z}\delta\,\mathrm{d}z;$$
$$c_{jk}=\int_0^H \frac{\mathrm{d}\phi_j}{\mathrm{d}z}\psi_k\delta\,\mathrm{d}z;\; c_{hi}=\int_0^H \psi_h\frac{\mathrm{d}\phi_i}{\mathrm{d}z}\delta\,\mathrm{d}z;$$
$$t_{jk}=\int_0^H \phi_j\frac{\mathrm{d}\psi_k}{\mathrm{d}z}\delta\,\mathrm{d}z;\; t_{hi}=\int_0^H \frac{\mathrm{d}\psi_h}{\mathrm{d}z}\phi_i\delta\,\mathrm{d}z \quad (3)$$

and

$$p_i=p(x,t)\phi_i(0)+\int_0^H p(x,z,t)\phi_i\,\mathrm{d}z$$
$$q_h=q(x,t)\psi_h(0)+\int_0^H q(x,z,t)\psi_h\,\mathrm{d}z \quad (4)$$

where $p(x, t)$ and $q(x, t)$ are the external surface loads acting along x and z directions respectively. The body force components $p(x, z, t)$ and $q(x, z, t)$ in dynamic problems can be interpreted as the intertial forces and can be written as

$$p(x,z,t)=-\frac{\rho\delta}{g^*}\frac{\partial^2 u}{\partial t^2};\; q(x,z,t)=-\frac{\rho\delta}{g^*}\frac{\partial^2 w}{\partial t^2} \quad (5)$$

where ρ is the unit weight of the material of the body and g^* is the acceleration due to gravity.

The formulation for the plane-strain case is essentially the same but the elastic constants E, ν have to be replaced by E_0, ν_0 where

$$E_0=E/(1-\nu^2) \quad \text{and} \quad \nu_0=\nu/(1-\nu) \quad (6)$$

The stresses σ_x and τ_{xz} at any cross-section x = constant can now be represented by '$m + n$' independent generalised forces as [5]

$$T_j(x, t) = \int_0^H \sigma_x \phi_j \delta \,dz = \frac{E}{(1-\nu^2)} \left\{ \sum_{i=1}^{m} a_{ji} U_i' + \sum_{k=1}^{n} t_{jk} W_k \right\} (i, j = 1, 2 \ldots\ldots m) \tag{7a}$$

$$S_h(x, t) = \frac{E}{2(1+\nu)} \left\{ \sum_{i=1}^{m} c_{hi} U_i + \sum_{k=1}^{n} r_{hk} W_k' \right\} (h, k = 1, 2 \ldots\ldots n). \tag{7b}$$

$T_j(x, t)$ and $S_h(x, t)$ are referred to as generalised longitudinal and shearing forces respectively. The required $2(m + n)$ boundary conditions to solve Eqns. (2), can now be prescribed in terms of these generalised forces considering the foundation medium to be an elastic body the above equations can be utilised directly to solve the beam foundation interaction problem.

3. DYNAMICS OF BEAM-FOUNDATION SYSTEM

An Euler-Bernoulli beam lying on an elastic foundation subjected to a dynamic load $q^*(x, t)$ is shown in Fig. 2. The differential equation for transverse vibration of the beam can be written as

$$E_b I \frac{\partial^4 w_0}{\partial x^4} = q^*(x, t) - q(x, t) - m_b \frac{\partial^2 w_0}{\partial t^2} \tag{8}$$

in which $E_b I$, w_0 and m_b are the modulus of rigidity, vertical deflection and mass per unit length of the beam. m_b can be expressed as

$$m_b = \rho_b \delta h / g^* \tag{9}$$

where ρ_b is the unit weight of the beam material and h is the height of the beam. Under plane-strain conditions I in Eq. (8) has to be replaced by J which is given by

$$J = I/(1 - \nu_b^2) \tag{10}$$

ν_b is the Poisson's ratio of the material of the strip.

Assuming compatibility at the interface of the beam-foundation system, from Eq. (1),

$$w_0 = \sum_{k=1}^{n} W_k(x, t) \psi_k(0) \tag{11}$$

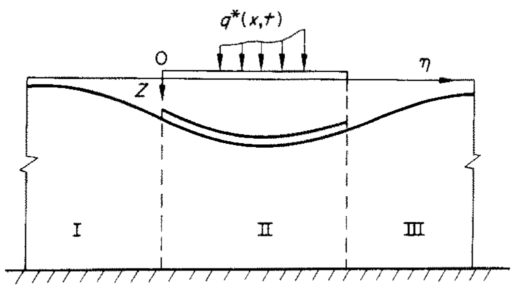

Fig. 2. Finite beam on elastic foundation.

From Eqns. (8), (11), (4) and (5)

$$q_h = q^*(x,t)\psi_h(0) - \sum_{k=1}^{n} E_b I \frac{\partial^4 W_k}{\partial x^4}\psi_k(0)\psi_h(0) - \sum_{k=1}^{n} m_b \frac{\partial^2 W_k}{\partial t^2}\psi_k(0)\psi_h(0) - \int_0^H \left\{\sum_{k=1}^{n} \frac{\rho\delta}{g^*}\frac{\partial^2 W_k}{\partial t^2}\psi_k(z)\psi_h(z)\,dz\right\}$$

$$(h = 1, 2, \ldots .n). \tag{12a}$$

From Eq. (5)

$$p_j = p(x,t)\phi_j(0) - \int_0^H \left\{\sum_{i=1}^{m} \frac{\rho\delta}{g^*}\frac{\partial^2 U_i}{\partial t^2}\phi_i(z)\phi_j(z)\,dz\right\}$$

$$(j = 1, 2 \ldots .m). \tag{12b}$$

Using Eqns. (12) and (3), Eqns. (2) can be written as

$$\sum_{i=1}^{m} a_{ij}U_i'' - \frac{(1-\nu)}{2}\sum_{i=1}^{m} b_{ji}U_i + \sum_{k=1}^{n}\left\{\nu t_{jk} - \frac{(1-\nu)}{2}c_{jk}\right\}W_k' + \frac{(1-\nu^2)}{E}\left\{p(x,t)\phi_j(0) - \frac{\rho}{g^*}\sum_{i=1}^{m} a_{ij}\frac{\partial^2 U_i}{\partial t^2}\right\} = 0$$

$$(j = 1, 2, \ldots .m) \tag{13a}$$

and

$$-\sum_{i=1}^{m}\left\{\nu t_{hi} - \frac{(1-\nu)}{2}c_{hi}\right\}U_i' + \frac{(1-\nu)}{2}\sum_{k=1}^{n} r_{hk}W_k'' - \sum_{k=1}^{n} s_{hk}W_k + \frac{(1-\nu^2)}{E}\left\{q^*(x,t)\psi_h(0) - \sum_{k=1}^{n} E_b I\frac{\partial^4 W_k}{\partial x^4}\psi_k(0)\psi_h(0) - \sum_{k=1}^{n} m_b\frac{\partial^2 W_k}{\partial t^2}\psi_k(0)\psi_h(0) - \frac{\rho}{g^*}\sum_{k=1}^{n} r_{hk}\frac{\partial^2 W_k}{\partial t^2}\right\} = 0$$

$$(h = 1, 2, \ldots ..n). \tag{13b}$$

Eqns. (13) are the general governing equations for the dynamic beam–foundation interaction problem.

4. GENERAL EQUATIONS IN TERMS OF AUXILIARY FUNCTIONS

As the beam has been assumed to transfer only vertical forces to the foundation, $p(x,t)$ in Eq. (13a) will be zero. Choosing one function each for $\phi(z)$ and $\psi(z)$ (i.e. $m = n = 1$) such that $\phi_1(0) = \psi_1(0) = 1$, and taking the displacement components to be proportional to $e^{i\omega t}$ (where ω is the operating frequency) for steady state response of the beam foundation system, in the absence of any other surface loads beyond the beam, Eqns. (13) can be expressed as

$$L_{11}U_1 + L_{12}W_1 = 0 \tag{14a}$$

$$L_{21}U_1 + L_{22}W_1 = -\left(\frac{1-\nu^2}{E}\right)q^* \tag{14b}$$

where U_1, W_1 and q^* are now functions of x only (as the responses and the external load are proportional to $e^{i\omega t}$). In Eqns. (14) L_{11}, L_{12}, L_{21} and L_{22} are differential operators given by

$$L_{11} = D_1\frac{d^2}{dx^2} + D_2; \quad L_{12} = D_3\frac{d}{dx}$$

$$L_{21} = -D_3\frac{d}{dx}; \quad L_{22} = D_4\frac{d^4}{dx^4} + D_5\frac{d^2}{dx^2} + D_6 \tag{15}$$

in which

$$D_1 = a_{11}; \quad D_2 = -\left\{\frac{(1-\nu)}{2}b_{11} - k_1^2a_{11}\right\};$$

$$D_3 = \left\{\nu t_{11} - \frac{(1-\nu)}{2}c_{11}\right\}; \quad D_4 = -\frac{(1-\nu^2)}{E}E_bI; \tag{16}$$

$$D_5 = \frac{(1-\nu)}{2}r_{11}; \quad D_6 = -(s_{11} - k_1^2r_{11} - k_2^2);$$

where a_{11}, b_{11} etc. are given by Eqn. (3) and

$$k_1^2 = \frac{(1-\nu^2)}{E}\frac{\rho\omega^2}{g^*} = a_0^2/L^2$$

and

$$k_2^2 = \frac{(1-\nu^2)}{E}m_b\omega^2 = b_o^2(\delta/L) \tag{17}$$

in which

$$a_0 = \omega L\sqrt{\frac{\rho}{g^*}\frac{(1-\nu^2)}{E}} \tag{18a}$$

$$b_0 = \omega L\sqrt{\frac{m_b}{\delta L}\frac{(1-\nu^2)}{E}} \tag{18b}$$

$$L = \left\{\frac{2E_bI(1-\nu^2)}{E\delta}\right\}^{1/3} \tag{18c}$$

a_0, b_0 and L can be referred to as dimensionless frequency ratio, dimensionless mass ratio and characteristic length respectively.

Equations (14) can now be uncoupled by choosing an auxiliary function F such that

$$U_1 = -L_{12}F = -D_3\left(\frac{dF}{dx}\right) \tag{19a}$$

and

$$W_1 = L_{11}F = D_1\frac{d^2F}{dx^2} + D_2F \tag{19b}$$

whereby Eqn. (14a) is automatically satisfied and Eqn. (14b) gives

$$\frac{d^6F}{d\eta^6}+\bar{f}_1\frac{d^4F}{d\eta^4}+\bar{f}_2\frac{d^2F}{d\eta^2}+\bar{f}_3F=\bar{f}_4q^* \tag{20}$$

in which $\eta = (x/L)$ = dimensionless parameter (21a)

$$\bar{f}_1=\left(\frac{D_1D_5+D_2D_4}{D_1D_4}\right)L^2;\quad \bar{f}_2=\left(\frac{D_1D_6+D_2D_5+D_3^2}{D_1D_4}\right)L^4 \tag{21b}$$

$$\bar{f}_3=\frac{(D_2D_6)}{D_1D_4}L^6;\quad \bar{f}_4=\frac{L^6}{E_bIa_{11}}. \tag{21c}$$

Eqn. (20) is the final equation governing the dynamic responses of the beam–foundation system.

With the generalised displacements U_1 and W_1 and generalised slope $W_1' = (1/L)W_1'(\eta)$, there correspond the generalised longitudinal force T_1, and generalised shearing force N_1 (which is distinct from the beam shear Q) given by

$$N_1 = Q + S_1 \tag{22}$$

where S_1 is the generalised transverse shear in the foundation. T_1 and S_1 can be obtained from Eqn. (7). In addition to these quantities, bending moment M of the beam and the soil reaction q at the interface (from Eqns. (2b) and (5)) have also to be defined. Using Eqn. (19), these can be defined as

$$U_1 = -D_3(dF/dx) = d_{U1}(dF/d\eta) \tag{a}$$

$$W_1 = D_1(d^2F/dx^2) + D_2F = d_{W1}(d^2F/d\eta^2) + d_{W2}F \tag{b}$$

$$W_1' = (dW_1/dx) = d_{WP1}(d^3F/d\eta^3) + d_{WP2}(dF/d\eta) \tag{c}$$

$$Q = -E_bI(d^3W_1/dx^3) = d_{Q1}(d^5F/d\eta^5) + d_{Q2}(d^3F/d\eta^3) \tag{d}$$

$$M = -E_bI(d^2W_1/dx^2) = d_{M1}(d^4F/d\eta^4) + d_{M2}(d^2F/d\eta^2) \tag{e}$$

$$T_1 = \frac{E}{(1-\nu^2)}\left(a_{11}\frac{dU_1}{dx} + \nu t_{11}W_1\right) = d_{T1}(d^2F/d\eta^2) + d_{T2}F \qquad (23) \tag{f}$$

$$S_1 = \frac{E}{2(1+\nu)}\left(c_{11}U_1 + r_{11}\frac{dW_1}{dx}\right) = d_{S1}(d^3F/d\eta^3) + d_{S2}(dF/d\eta) \tag{g}$$

$$N_1 = Q + S_1 = d_{N1}(d^5F/d\eta^5) + d_{N2}(d^3F/d\eta^3) + d_{N3}(dF/d\eta) \tag{h}$$

$$q = \frac{E}{(1-\nu^2)}\left[-\left\{\nu t_{11} - \frac{(1-\nu)}{2}c_{11}\right\}\frac{dU_1}{dx} + \frac{(1-\nu)}{2}r_{11}\frac{d^2W_1}{dx^2} - (s_{11} - k_1^2 r_{11})W_1\right]$$

$$= d_{q1}(d^4F/d\eta^4) + d_{q2}(d^2F/d\eta^2) + d_{q3}F \tag{i}$$

where

$$
\begin{aligned}
&d_{U1} = (-D_3/L);\quad d_{W1} = (D_1/L^2);\quad d_{W2} = D_2;\quad d_{WP1} = (D_1/L^3);\\
&d_{WP2} = (D_2/L);\quad d_{Q1} = -D_1(E_bI/L^5);\quad d_{Q2} = -D_2(E_bI/L^3);\\
&d_{M1} = -D_1(E_bI/L^4);\quad d_{M2} = -D_2(E_bI/L^2);\quad d_{T1} = \frac{E}{(1-\nu^2)}\frac{(\nu t_{11}D_1 - a_{11}D_3)}{L^2}\\
&d_{T2} = D_2\nu t_{11}E/(1-\nu^2);\quad d_{S1} = \frac{E}{2(1+\nu)}(r_{11}D_1/L^3);\\
&d_{S2} = \frac{E}{2(1+\nu)}(r_{11}D_2 - c_{11}D_3)/L;\quad d_{N1} = d_{Q1} = -E_bID_1/L^5 \qquad (24)\\
&d_{N2} = d_{Q2} + d_{S1} = \left\{\frac{E}{2(1+\nu)}r_{11}D_1 - E_bID_2\right\}\Big/L^3;\\
&d_{N3} = d_{S2} = \frac{E}{2(1+\nu)}(r_{11}D_2 - c_{11}D_3)/L;\quad d_{q1} = \frac{-E}{(1-\nu^2)}(D_5D_1/L^4),\\
&d_{q2} = -\frac{E}{(1-\nu^2)}\{D_3^2 + D_5D_2 - D_1(s_{11} - k_1^2r_{11})\}/L^2;\\
&d_{q3} = \frac{E}{(1-\nu^2)}D_2(s_{11} - k_1^2r_{11})
\end{aligned}
$$

5. DYNAMIC RESPONSES OF FINITE BEAMS ON ELASTIC FOUNDATIONS

The results of static responses of infinite beams and finite beams on generalised elastic foundations as idealised by the proposed foundation model [4, 5] bring out the reliability of its performance. The above equations reduce to static equations by omitting the time variable, t, (or $\omega = 0$), i.e. k_1, k_2, a_0 and b_0 equal to zero identically. Essentially following the same approach applied to static problems [4, 5], the solutions to the dynamic responses of the beam–foundation system have been briefly presented below using the method of initial parameters while a detailed account of the method is discussed in references [4] and [5].

For the region II (Fig. 2), using the homogeneous solution of Eq. (20), the quantities U_1 to q given by Eqn. (23) can be expressed in terms of the values of the generalised parameters W_1, W_1', U_1, T_1, M_1 and N_1 at any initial section say $\eta = 0$, as [4].

$$
\begin{Bmatrix} W_1 \\ W_1' \\ U_1 \\ T_1 \\ M \\ N_1 \\ Q \\ q \end{Bmatrix} = \{R\} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{21} & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \\ K_{71} & K_{72} & K_{73} & K_{74} & K_{75} & K_{76} \\ K_{81} & K_{82} & K_{83} & K_{84} & K_{85} & K_{86} \end{bmatrix} \begin{Bmatrix} (W_1)_0 \\ (W_1')_0 \\ (U_1)_0 \\ (T_1)_0 \\ (M)_0 \\ (N_1)_0 \end{Bmatrix} = [K]\{I_p\} \qquad (25)
$$

K_{11} to K_{86} are called influence functions and $\{I_p\}$ are called initial parameters. $[K]$ forms a matrix for direct linear transformation of $\{I_p\}$ into $\{R\}$. From the homogeneous solution of Eqn. (20) and Eqns. (23), all the components $[K]$ are completely known. Hence if the initial parameters

$\{I_p\}$ are known the problem can be considered to be completely solved. In general, if at any section $\eta = s$, the parameters $(W_1)_s$, $(W_1')_s$, $(U_1)_s$, $(T_1)_s$, $(M)_s$, and $(N_1)_s$ are known, they can be taken as initial parameters. Then at any section η, the quantities $(W_1)_\eta$, $(W_1')_\eta$, $(U_1)_\eta$, $(T_1)_\eta$, $(M)_\eta$, $(N_1)_\eta$, $(Q)_\eta$, and $(q)_\eta$ can be determined by the same influence functions K_{11} to K_{86} with the argument $(\eta - s)$ (since the section at which the quantities are to be evaluated is at a distance of $(\eta - s)$ from the initial section $\eta = s$), provided the homogeneous equation of bending is valid between these two sections, which means that there are no external loads and (or) moments etc. applied in between the two-sections.

In case some external loads and (or) moments are applied, their effect will be to change the values of some of the parameters W_1, W_1', U_1, T_1, M and N_1 (at the section where these external influences have been applied) and hence have to be taken into account in writing the solutions beyond the section where the external influences have been applied. Hence in the most general case of a beam on an elastic foundation subjected to arbitrary external influences (however, in the dynamic case, the operating frequency, ω, has to be the same, while they can be discrete concentrated or distributed loads and (or) moments etc.), the non-homogeneous solutions can be written as

$$\{R\} = [K]\{I_p\} - \{\alpha\} \tag{26a}$$

where $\{R\}$, $[K]$, $\{I_p\}$ are defined by Eqn. (25) and

$$\{\alpha\}^T = \{F_{W1}, F_{WP1}, F_{U1}, F_{T1}, F_M, F_{N1}, F_Q, F_q\} \tag{26b}$$

in which the superscript "T" stands for the transposition of the matrix.

$\{\alpha\}$ can be obtained knowing $[K]$ and the èxternally applied loads and (or) moments as discussed in detail in Refs. [4] and [5] and represents the non-homogeneous part. From Eqns. (26), the required dynamic responses can be obtained at any section of the beam provided the initial parameters $\{I_p\}$ are known. These can be evaluated from the boundary conditions at the two ends of the beam.

6. BONDARY CONDITIONS

Three practically important cases of end conditions of the beam can be studied.

6.1 *Free end*

In this case, the continuity conditions for the beam and the elastic foundation have to be accounted for, in prescribing the boundary conditions. Accordingly, W_1, U_1, T_1 and N_1 are seen to be compatible with the corresponding quantities of the foundation at each free end of the beam. Also the bending moment, M at each free end is zero.

6.2 *Simply supported end*

Boundary conditions in this case are $W_1 = U_1 = M = 0$ at the simply supported end.

6.3 *Fixed or built-in end*

In this case of a fixed end, $W = W_1' = U_1 = 0$.

Any combination of these end conditions can be similarly analysed.

7. FINITE BEAM WITH FREE ENDS

Referring to Fig. 2 of a finite beam on elastic foundation subjected to arbitrary external dynamic loads, the responses of part II of the beam–foundation system are governed by Eq. (20), while the responses of the free part I and III of the foundation (beyond the beam) are governed by Eqns. (2). As already noted, with one function each for $\phi(z)$ and $\psi(z)$ i.e. ($m = n = 1$) such that $\phi_1(0) = \psi_1(0) = 1$, and taking the displacements proportional to $e^{i\omega t}$, Eqns. (2) get uncoupled by substituting for U_1 and W_1 from Eqns. (19). Using the auxiliary function F_f in the place of F to designate the solution for the free parts of the foundation, (and noting that there are no other external surface loads on the foundation) the resulting equation can be obtained as

$$\frac{d^4F_f}{d\eta^4} + \bar{C}_1\frac{d^2F_f}{d\eta^2} + \bar{C}_2F_f = 0 \tag{27}$$

where

$$\bar{C}_1 = (D_3^2 + D_2D_5 + D_7D_1)L^2/(D_1D_5) \tag{28}$$

$$\bar{C}_2 = D_2D_7L^4/(D_1D_5)$$

where D_1 to D_6 are given by Eqns. (16) and

$$D_7 = -(s_{11} - k_1^2a_{11}) \tag{29}$$

and L is the characteristic length given by Eqn. (18c), η is the dimensionless parameter given by Eqn. (21a).

The solution of Eqn. (27) depends on the nature of characteristic roots. Only three cases of solutions depending on the range of parameters involved are discussed below while all other cases are dealt in Appendix A. While no results are reported Rades [6] also fixes the ranges over which physically feasible solutions for the free part of the foundation are possible.

Case 1

The characteristic roots of Eqn. (27) are real and unequal.

Keeping in view the boundedness of the responses as $\eta \to \infty$, the physically feasible solutions can be written as

$$F_f(\eta) = A_1e^{\bar{m}_1\eta} + A_2e^{\bar{m}_2\eta} \text{ for } \eta \le 0 \tag{30a}$$

$$= A_3e^{-\bar{m}_1\eta} + A_4e^{-\bar{m}_2\eta} \text{ for } \eta \ge 0 \tag{30b}$$

in which $\bar{m}_1$, $\bar{m}_2$ are positive quantities given by

$$\bar{m}_{1,2} = \{(-\bar{C}_1 \pm \sqrt{\bar{C}_1^2 - 4\bar{C}_2})/2\}^{1/2} \tag{31}$$

and A_1 to A_4 are arbitrary constants.

Case 2

The characteristic roots of Eqn. (27) are real and equal. The physically feasible solutions in this case can be written as

$$F_f(\eta) = (A_1 + A_2\eta)\,e^{\bar{m}_0\eta} \text{ for } \eta \le 0 \tag{32a}$$

$$= (A_3 + A_4\eta)\,e^{-\bar{m}_0\eta} \text{ for } \eta \ge 0 \tag{32b}$$

where $\bar{m}_0$ is given by

$$\bar{m}_0 = (-\bar{C}_1/2)^{1/2} \tag{33}$$

and A_1 to A_4 are arbitrary constants.

Case 3

The characteristic roots of Eqn. (27) are complex conjugate with non-zero real part.

This happens if $(\bar{C}_1^2 - 4\bar{C}_2)$ is negative and $\bar{C}_1$ is not equal to zero. For all practically possible ranges of parameters it is noted that $\bar{C}_1^2 - 4\bar{C}_2$ is always positive and hence, while physically feasible solutions can be obtained in this case, they seem to be of academic interest only.

In the examples illustrated below, the nature of the roots are observed to belong to either case 1 or case 2. However, for certain ranges of parameters, other types of characteristic roots for the Eqn. (27) are possible and those are briefly discussed in Appendix A.

Using these solutions, the displacements and forces W_1, U_1, T_1, and S_1 of the free parts of the foundation (parts I and III in Fig. 2) can be evaluated from Eqns. (23a, b, f and g) and it is to be noted that the function F to be substituted in these equations is the homogeneous solution F_f of Eqn. (27) as given by Eqns. (30) or (32) or other appropriate forms depending on the characteristic roots of Eqn. (27). However for part II of the foundation beneath the beam, W_1, U_1, T_1 and N_1 can be obtained (Eqns. (25) and (26)) from the same equations, function F being the solution of Eqn. (20).

From the continuity conditions of the forces and displacements, the boundary conditions at each free end can be written as (depending on the end)

$$\begin{aligned} &(W_1)_{II} = (W_1)_I \quad \text{or} \quad (W_1)_{III}; \quad (U_1)_{II} = (U_1)_I \quad \text{or} \quad (U_1)_{III} \\ &(T_1)_{II} = (T_1)_I \quad \text{or} \quad (T_i)_{III}; \quad (N_1)_{II} = (S_1)_I \quad \text{or} \quad (S_1)_{III} \end{aligned} \tag{34}$$

and bending moment $M = 0$.

Using the appropriate solution for F_f (as given by Eqns. (30) and 32)) in Eqns. (23a, b, f and g), and taking the origin (Fig. 2) at the left hand end of the beam the initial parameters can now be expressed from Eqns. (34) as

$$\begin{Bmatrix} (W_1)_0 \\ (W_1')_0 \\ (U_1)_0 \\ (T_1)_0 \\ (M)_0 \\ (N_1)_0 = (S_1)_0 \end{Bmatrix} = \{I_p\} = \begin{Bmatrix} A_1H_{W1} + A_2H_{W2} \\ (W_1')_0 \\ A_1H_{U1} + A_2H_{U2} \\ A_1H_{T1} + A_2H_{T2} \\ 0 \\ A_1H_{S1} + A_2H_{S2} \end{Bmatrix} \tag{35}$$

Taking "$2l$" as the length of the beam, at the other end $\eta = 2l/L$, in order to make use of the continuity and boundary conditions given by Eqns. (34), the corresponding quantities can be expressed depending on the nature of solution F_f as follows:

$$\begin{Bmatrix} (W_1)_{2l/L} \\ (U_1)_{2l/L} \\ (T_1)_{2l/L} \\ (M)_{2l/L} \\ (N_1)_{2l/L} = (S_1)_{2l/L} \end{Bmatrix} = \begin{Bmatrix} (A_3H_{W1} \pm A_4H_{W2}) \\ -(A_3H_{U1} \pm A_4H_{U2}) \\ (A_3H_{T1} \pm A_4H_{T2}) \\ 0 \\ -(A_3H_{S1} \pm A_4H_{S2}) \end{Bmatrix} \tag{36}$$

In the right hand side expressions in Eqn. (36), the upper positive sign should be taken in case the solutions of the free part of the foundation, F_f, are of the form given by Eqns. (30) and the lower negative sign in case F_f is of the form given by Eqns. (32). The Expressions H_{W1} etc. also depend on the nature of solution of F_f and can be obtained as below.

Case 1

F_f is of the form given by Eqns. (30).

$$\begin{aligned} &H_{W1} = d_{W1}\bar{m}_1^2 + d_{W2}; \quad H_{W2} = d_{W1}\bar{m}_2^2 + d_{W2}; \quad H_{U1} = d_{U1}\bar{m}_1; \\ &H_{U2} = d_{U1}\bar{m}_2; \quad H_{T1} = d_{T1}\bar{m}_1^2 + d_{T2}; \quad H_{T2} = d_{T1}\bar{m}_2^2 + d_{T2}; \\ &H_{S1} = d_{S1}\bar{m}_1^3 + d_{S2}\bar{m}_1; \quad H_{S2} = d_{S1}\bar{m}_2^3 + d_{S2}\bar{m}_2; \end{aligned} \tag{37}$$

where d_{W1} etc. are given by Eqns. (24) and $\bar{m}_1$, $\bar{m}_2$ are given by Eqns. (31).

Case 2

F_f is of the form given by Eqns. (32).

$$\begin{aligned} &H_{W1} = d_{W1}\bar{m}_0^2 + d_{W2}; \quad H_{W2} = 2d_{W1}\bar{m}_0; \quad H_{U1} = d_{U1}\bar{m}_0; \\ &H_{U2} = d_{U1}; \quad H_{T1} = d_{T1}\bar{m}_0^2 + d_{T2}; \quad H_{T2} = 2d_{T1}\bar{m}_0; \\ &H_{S1} = d_{S1}\bar{m}_0^3 + d_{S2}\bar{m}_0; \quad H_{S2} = 3d_{S1}\bar{m}_0^2 + d_{S2} \end{aligned} \tag{38}$$

in which d_{W1} etc. are given by Eqns. (24) and $\bar{m}_0$ is given by Eqn. (33).

Similar expressions can be obtained in case of other feasible forms of the solution for F_f.

Finally from the general solutions given by Eqn. (26) for the steady state dynamic responses of beams on generalised elastic foundations subjected to arbitrary external influences, and from Eqns. (35) and (36), the required number of equations to solve for the hitherto unknown quantities A_1, A_2, A_3, A_4 and $(W_1')_0$ can be written as

$$\begin{Bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ K_{31} & K_{32} & K_{33} & K_{34} & K_{35} & K_{36} \\ K_{41} & K_{42} & K_{43} & K_{44} & K_{45} & K_{46} \\ K_{51} & K_{52} & K_{53} & K_{54} & K_{55} & K_{56} \\ K_{61} & K_{62} & K_{63} & K_{64} & K_{65} & K_{66} \end{Bmatrix}_{\eta = \frac{2l}{L}} \begin{Bmatrix} A_1H_{W1} + A_2H_{W2} \\ (W_1')_0 \\ A_1H_{U1} + A_2H_{U2} \\ A_1H_{T1} + A_2H_{T2} \\ 0 \\ A_1H_{S1} + A_2H_{S2} \end{Bmatrix}$$

$$- \begin{Bmatrix} A_3H_{W1} \pm A_4H_{W2} \\ -(A_3H_{U1} \pm A_4H_{U2}) \\ A_3H_{T1} \pm A_4H_{T2} \\ 0 \\ -(A_3H_{S1} \pm A_4H_{S2}) \end{Bmatrix} = \begin{Bmatrix} F_{W1} \\ F_{U1} \\ F_{T1} \\ F_M \\ F_{N1} \end{Bmatrix}_{\eta = 2l/L.} \tag{39}$$

In the left hand side expressions of Eqn. (39), as indicated earlier, the upper positive sign should be taken in case F_f is given by Eqn. (30) and lower negative sign in case F_f is given by Eqn.

(32). The components of $[K]$ and the non-homogeneous components F_{W1}, F_{U1} etc. at $\eta = 2l/L$ are completely known as explained above in Section 5. Thus Eqns. (39) result in five equations in the five unknowns, A_1, A_2, A_3, A_4 and $(W_1')_0$ which can be uniquely solved. Knowing these, the initial parameters, $\{I_p\}$ can be obtained as given by Eqns. (35). Once $\{I_p\}$ are known, the solutions for region II (Fig. 2) of the beam–foundation system are completely known at any η, as given by Eqns. (26), while the solutions for the free parts I and III (Fig. 2) of the foundation can be obtained from Eqns. (23) by the appropriate substitution of F_f as given by Eqns. (30) and (32). (The solutions for F_f are completely known since the constants A_1 to A_4 can be solved from Eqns. (39)).

8. SIMPLY SUPPORTED AND FIXED BEAMS

In these cases the foundations do not extend beyond the beam ends and prescribing the necessary boundary conditions as mentioned in Section 6, solutions can be obtained [5] using Eqns. (26).

9. ILLUSTRATIONS AND RESULTS

While the solutions presented in the above analysis are quite general a few cases have been illustrated below.

Choosing the displacement distribution functions as

$$\phi_1(z) = \psi_1(z) = e^{-\gamma z} \tag{40}$$

and defining the dimensionless parameter,

$$\lambda = \gamma L. \tag{41}$$

The responses have been presented for two types of loading.

Case 1

Concentrated dynamic load $P = P_0 e^{i\omega t}$ acting at the mid span of the beam.

In this case the responses have been expressed as

$$W_1(\eta) = \frac{(1-\nu^2)P_0 e^{i\omega t}}{E\delta}\bar{W}; \quad M(\eta) = lP_0 e^{i\omega t}\bar{M};$$

$$Q(\eta) = P_0 e^{i\omega t}\bar{Q}; \quad q(\eta) = \frac{P_0 e^{i\omega t}}{l}\bar{q} \tag{42}$$

in which "l" is the half length of the beam.

Case 2

Uniformly distributed dynamic load $q_u = q_0 e^{i\omega t}$ acting on the entire length of the beam.

In this case the responses have been expressed as

$$W_1(\eta) = \frac{2l(1-\nu^2)}{E\delta} q_0 e^{i\omega t}\bar{W}; \quad M(\eta) = l^2 q_0 e^{i\omega t}\bar{M}$$

$$Q(\eta) = lq_0 e^{i\omega t}\bar{Q}; \quad q(\eta) = q_0 e^{i\omega t}\bar{q}. \tag{43}$$

For a few typical values of λ, l/L, ν, a_0 and b_0, the dimensionless dynamic responses $\bar{W}$, $\bar{M}$, $\bar{Q}$ and $\bar{q}$ have been graphically shown in Figs. 3 to 6.

In Figs. 3(a) and (3b), results obtained from the present analysis are compared with those presented by Rades[6] neglecting the foundation inertia (which in the present analysis is equivalent to putting $a_0 = 0$). Both the cases of Pasternak's model and Winkler's model are compared. To convert the parameters used by Rades[6] into those used in the present investigation, the mathematical equivalence of Vlasov and Leontev's model[3] with that of Pasternak's model is made use of and the parameters λ, ν and l/L are so chosen as to yield the values used by Rades for computational purposes. By so doing, the dynamic responses of beams on Vlasov and Leontev's foundation model using the above values of λ, ν and l/L will turn out to be the same as reported by Rades[6]. The ratio of dynamic deflection at the centre of the beam, W_C, to the static deflection at the centre of the beam, W_C^* is graphically shown in Fig. 3(a) as a function of Ω, where Ω can be worked out as

$$\Omega = \sqrt{2/\lambda}(b_0) \tag{44}$$

Similarly the ratio of dynamic moment at the centre of the beam, M_C, to the static moment at the centre of the beam, M_C^* is shown in Fig. 3(b) as a function of Ω. In these cases (Figs. 3(a) and 3(b)), $\lambda = 0{\cdot}5872$, $l/L = 1{\cdot}1420$, $\nu = 0{\cdot}3$ and $a_0 = 0$.

Dynamic responses $\bar{W}$, $\bar{M}$, $\bar{Q}$ and $\bar{q}$ (Eqns. (42)) of a beam with free ends, subjected to a concentrated dynamic load at centre, P, and taking into account the foundation inertia ($a_0 \neq 0$) have been presented in Figs. 4(a) to 4(d).

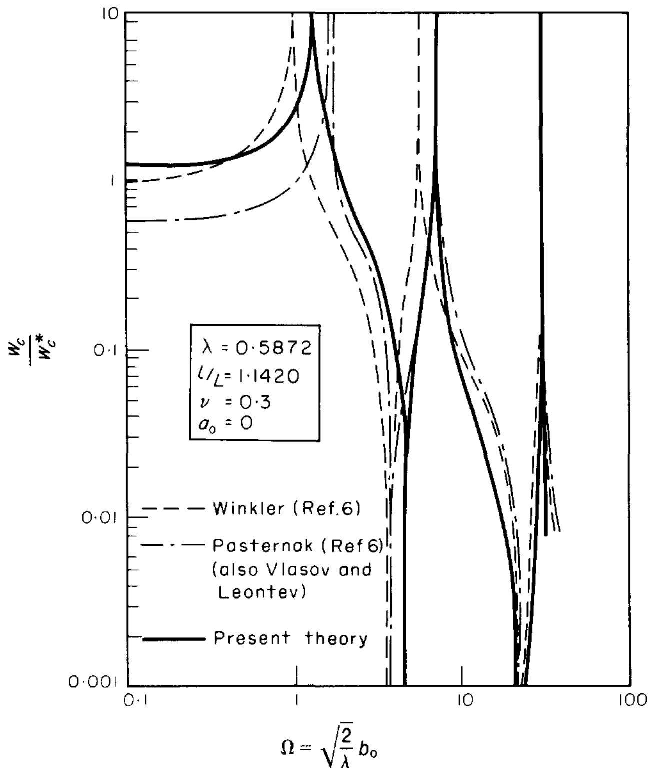

Fig. 3(a). Dimensionless dynamic deflection vs Ω.

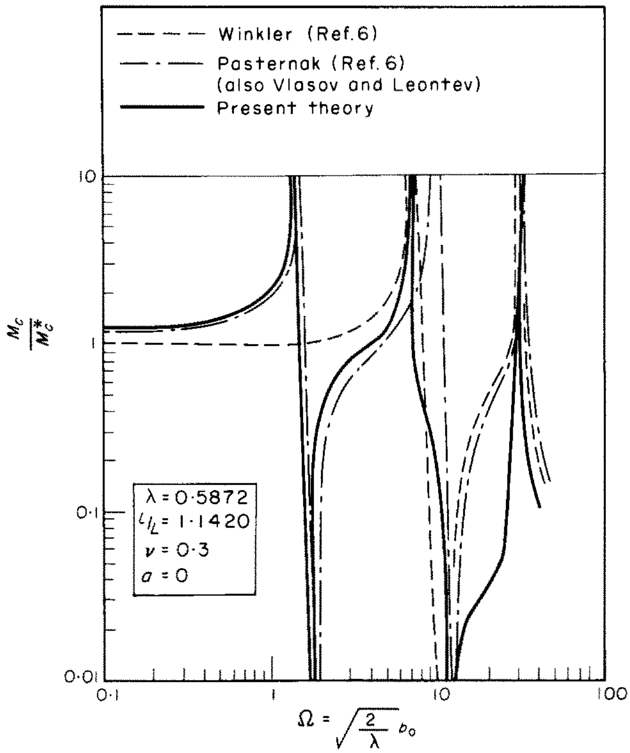

Fig. 3(b). Dimensionless dynamic moment vs Ω.

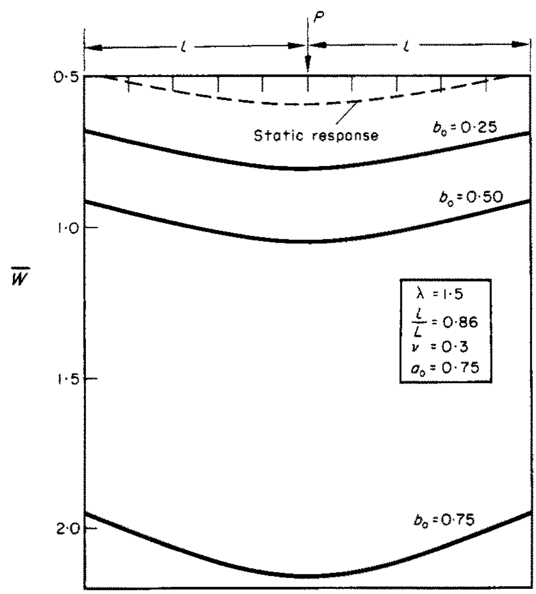

Fig. 4(a). Dynamic response $\bar{W}$ of a beam with free ends.

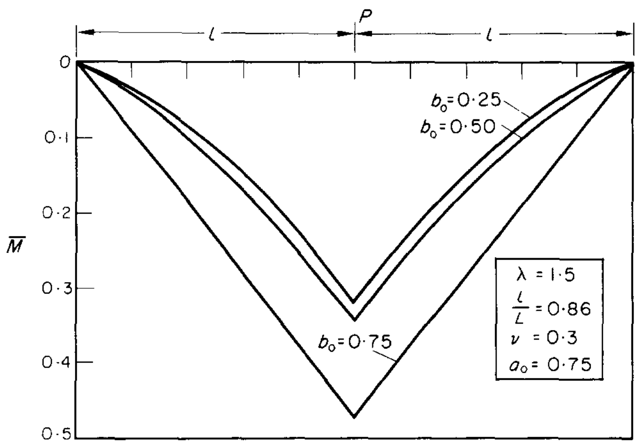

Fig. 4(b). Variation of $\bar{M}$ (dynamic case).

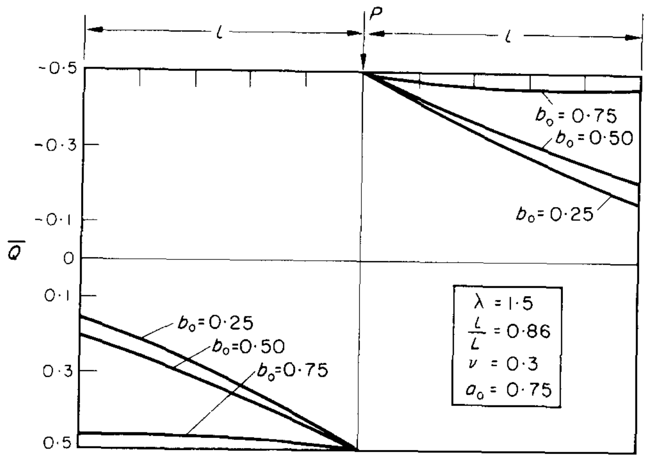

Fig. 4(c). Variation of $\bar{Q}$ (dynamic case).

The dynamic responses $\bar{W}$ and $\bar{M}$ (Eqns. (43)) of a beam subjected to a uniformly distributed dynamic load, q_u and taking foundation inertia also into account ($a_0 \neq 0$) are presented in Figs. 5(a) and 5(b). For the same case, the dimensionless dynamic contact pressure, $\bar{q}$ (Eqns. (43)) is compared for beams with different end conditions in Fig. 6, to estimate the effect of end conditions of the beam on the contact pressure distribution.

By putting $a_0 = 0$, and $b_0 = 0$ in the above analysis solutions to the static case can be obtained.

10. CONCLUSIONS

While the performance of the proposed model in static problems is brought out in Ref. [4], Figs. 3(a) and 3(b) show the comparative performance of the proposed generalised foundation model in dynamic analysis. The present responses are in general similar to those reported by Rades [6]. The present analysis gives higher values for deflections in the initial stages, the first natural frequency being in between the values obtainable using Winkler's foundation model and

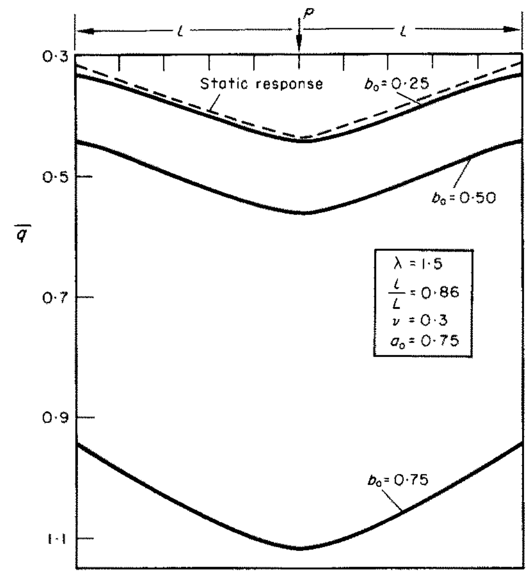

Fig. 4(d). Variation of $\bar{q}$ (dynamic case).

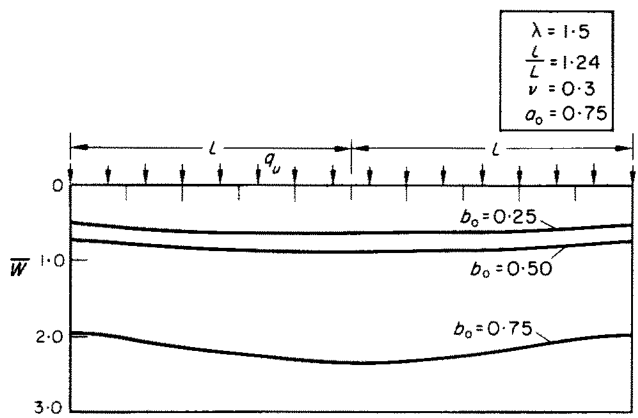

Fig. 5(a). Variation of $\bar{W}$ (dynamic case).

Pasternak foundation model (Fig. 3(a)). Regarding bending moments (Fig. 3(b)), the present analysis also reveals the insufficiency of the Winkler model. The dynamic analysis of beams on Winkler's foundation completely misses the first natural frequency [6], while the present analysis picks up the first natural frequency just as Pasternak foundation, the present value being slightly less than the one reported by Rades [6].

Accounting for the foundation inertia ($a_0 \neq 0$) a few results for the cases of concentrated and uniformly distributed dynamic loads have been presented in Figs. 4 and 5. The dependance of dynamic contact pressure distribution on the boundary conditions of the beam is shown in Fig. 6, taking into account the foundation inertia. As can be expected, the $\bar{q}$ values are higher in the case of beams with free ends when compared to beams with simply supported and fixed ends.

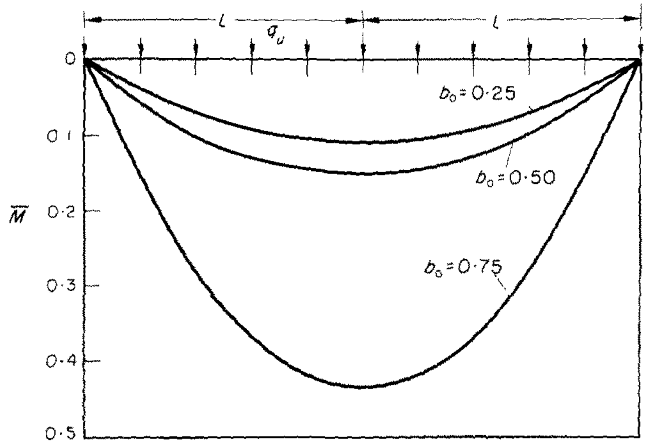

Fig. 5(b). Variation of $\tilde{M}$ (dynamic case).

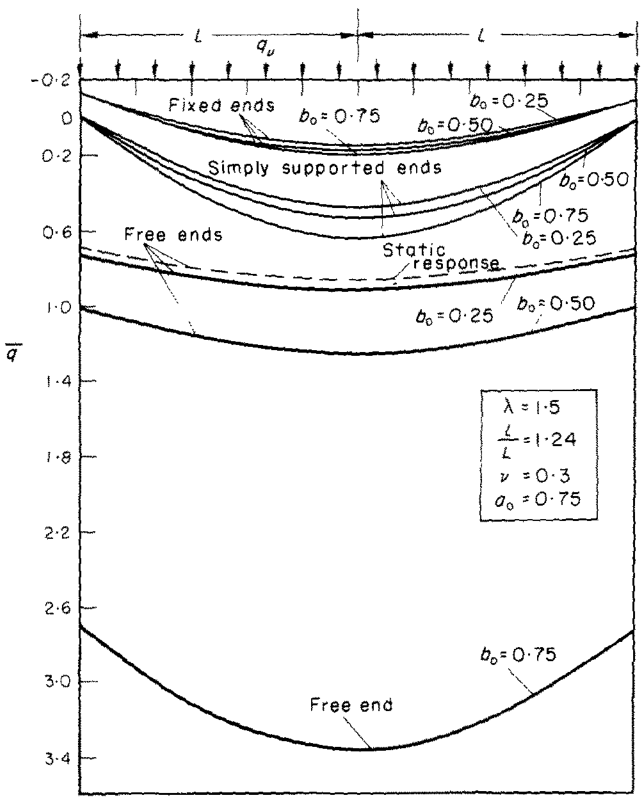

Fig. 6. Variation of $\bar{q}$ for beams with different end conditions (dynamic case).

The generalised foundation model presented in this investigation is flexible enough to take care of all the physical parameters and their variations that might be encountered in practice.

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APPENDIX A

Solutions F_f, *for the free part of foundation* (Eq. (27))

The forms of solution for F_f from Eq. (27) depend on the nature of the characteristic roots, which in turn depend on the values of certain physical parameters. The solutions thus obtained should be physically feasible in the sense that the responses should be bounded as $\eta \to \infty$. Rades[6] also fixed certain range of Ω, up to which only the solutions are feasible. Referring to Eq. (27):

1. $\nu = 0$, $a_0^2/\lambda^2 = 0$. (Foundation Inertia is neglected). The characteristic roots are of the type $\pm m_0$, $\pm m_0$. The physically feasible solutions are given by Eqns. (32).
2. $0 \le a_0^2/\lambda^2 < (1-\nu)/2$. ($a_0 = 0$ corresponds to zero foundation inertia). The characteristic roots are of the type $\pm m_1$, $\pm m_2$ and the physically feasible solutions are given by Eqns. (30).
3. $a_0^2/\lambda^2 = (1-\nu)/2$. Characteristic roots are of the type ± 0, $\pm m_1$.
4. $(1-\nu)/2 < a_0^2/\lambda^2 < 2(1-\nu)(1+2\nu)/(3-\nu)$. The roots are of the type, $\pm m_1$, $\pm i m_2$ where $i = \sqrt{-1}$.
5. $2(1-\nu)(1+2\nu)/(3-\nu) = a_0^2/\lambda^2$. The roots are of the type $\pm m_1$, $\pm i m_1$, where $i = \sqrt{-1}$.
6. $2(1-\nu)(1+2\nu)/(3-\nu) < a_0^2/\lambda^2 < 1$. Roots are $\pm m_1$, $\pm i m_2$, where $i = \sqrt{-1}$.
7. $a_0^2/\lambda^2 = 1$. The roots are of the type ± 0, $\pm i m_1$, where $i = \sqrt{-1}$.
8. $a_0^2/\lambda^2 > 1$. The characteristic roots are of the type $\pm i m_1$, $\pm i m_2$, where $i = \sqrt{-1}$.

Over all the practical ranges of parameters, it can be noted that the characteristic roots are never complex conjugate with non-zero real part.

In the first two cases the physically feasible solutions have been dealt in detail earlier as the examples quoted essentially belong to these categories. In the rest of the cases 3 to 8, while it is realised that the feasible solutions cannot be identified readily as in the first two cases, it seems logical to use the physical properties of symmetry of the problem and boundedness of responses in writing the feasible solutions. As an example, considering case 8, the general solution for F_f can be written as

$$F_f = B_1 \cos m_1\eta + B_2 \sin m_1\eta + B_3 \cos m_2\eta + B_4 \sin \mu_2\eta \tag{A1}$$

Considering the symmetry of the problem, it seems logical to write the feasible solutions as (by retaining the symmetric cosine terms only in the solution).

$$\begin{aligned} F_f &= B_1 \cos m_1\eta + B_3 \cos m_2\eta \text{ for } \eta \ge 0 \\ &= \bar{B}_1 \cos m_1\eta + \bar{B}_3 \cos m_2\eta \text{ for } \eta \le 0 \end{aligned} \tag{A2}$$

Such a situation is reported by Kenney[7] and Weitsman[8]. However they obtained the feasible solutions by incorporating a fictitious damping term in the differential equation, and identifying the forward wave and backward wave and then reducing the damping to zero. In their cases, the waves have been observed to be anti-symmetric for certain ranges of parameters.

Once the feasible solution for F_f can be obtained the rest of the analysis follows as already dealt with.